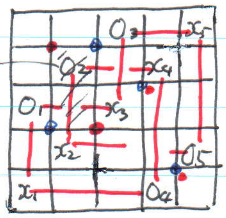
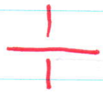
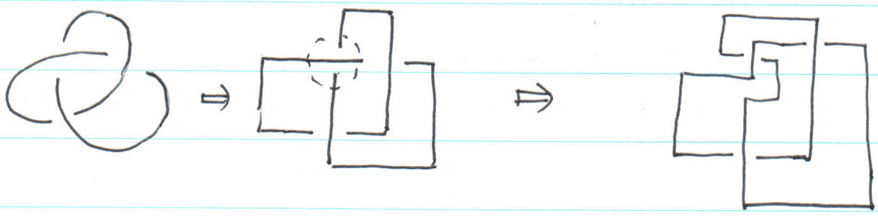
Manolescu

Grid diagram

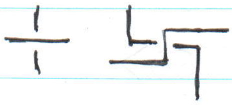
r 

 = diagram for a knot K

Lemma Every knot has a grid diagram



replace



Versions of knot Floer homology

G : grid diagram $\qquad \mathbb{F} = \mathbb{Z}/2$

$$S(G) = \{ \vec{x} = (x_{1\sigma(1)}, \dots, x_{n\sigma(n)}) \mid \sigma \in \mathfrak{S}_n \}$$

$$x_{ij} = (i, j)$$

$$C^-(G) = \mathbb{F}\langle S(G) \rangle [U_1, \dots, U_n]$$

$\partial^- : C^-(G) \circlearrowleft$

$$\partial^- \vec{x} = \sum_{\vec{y} \in S(G)} \sum_{\substack{r \in \mathrm{Rect}^\circ(\vec{x}, \vec{y}) \\ x_i(r) = 0 \ \forall i}} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \vec{y}$$

$$H_*(C^-(G), \partial^-) = HFK^-(K)$$

$\mathbb{F}[U]$-module $\quad U = U_i \ \forall i$

② $\hat{C}(G) = \mathbb{F}[U_2, \dots, U_n] \langle S(G) \rangle \quad$ set $U_1 = 0$

$$H_*(\hat{C}(G), \hat{\partial}) = \widehat{HFK}(K)$$

$\mathbb{F}$-v.sp. $\quad U_i = 0 \ \forall i$

finite dim'l

$$0 \to C^-(G) \xrightarrow{U_1} C^-(G) \to \hat{C}(G) \to 0$$

long exact sequence $\quad \to HFK^- \xrightarrow{U} HFK^- \to \widehat{HFK} \to \cdots$

Set $U_2 = 0$ $\quad \hat{\hat{C}}(G) = \mathbb{F}[U_3, \cdots, U_n]\langle S(G)\rangle \quad \hat{\widehat{HFK}}$

$$\widehat{HFK} \xrightarrow[0]{\overset{U_2}{=}} \widehat{HFK} \to \hat{\widehat{HFK}}$$

$\therefore \hat{\widehat{HFK}} = \widehat{HFK} \otimes V$

rank $V = 2$

③ $\tilde{C}(G)$ $\quad$ Set $U_1 = U_2 = \cdots = U_n = 0$

$$\tilde{C}(G) = \mathbb{F}\langle S(G)\rangle$$

$$\tilde{\partial}\vec{x} = \sum_{\vec{y}} \sum_{\substack{r \in Rect^\circ(\vec{x}, \vec{y}) \\ X_i(r) = 0 \ \forall i \\ O_i(r) = 0}} \vec{y}$$

just count completely empty rectangles

Result $\quad \widetilde{HFK} = \widehat{HFK} \otimes V^{n-1}$

not quite a knot invariant

④ Most complete theory

count all $r \in Rect^{(\circ)} \cdots \vec{x} \cap \mathrm{int}(r) = \phi$

allow $X_i(r), O_i(r) \neq 0$

$$C(G) = C^-(G) = \mathbb{F}[U_1, \cdots, U_n]\langle S(G)\rangle$$

$$\partial\vec{x} = \sum_{\vec{y}} \sum_{r \in Rect^\circ(\vec{x}, \vec{y})} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \vec{y}$$

It turns out $\quad H_*(C(G), \partial) = \mathbb{F}[U]$

for any knot

Recall Alexander, homological gradings on $S(G)$

If $r \in \mathrm{Rect}(\vec{x}, \vec{y})$

$$A(\vec{x}) - A(\vec{y}) = \sum_i x_i(r) - O_i(r)$$

$$M(\vec{x}) - M(\vec{y}) = 1 + 2\underset{\#(\vec{X} \cap \mathrm{Int}(r))}{\underset{\shortparallel}{G(r)}} - 2\sum_i O_i(r)$$

multiply by U_i changes A by -1, M by -2

∂ lowers M by 1

$r \in \mathrm{Rect}^{\circ}(\vec{x}, \vec{y}) \Rightarrow A(\vec{x}) - A(U_1^{O_1(r)} \cdots U_n^{O_n(r)} \vec{y})$

$$= A(\vec{x}) - A(\vec{y}) + \sum O_i(r) = \sum x_i(r) \geqq 0$$

$\therefore$ ∂ never increase A

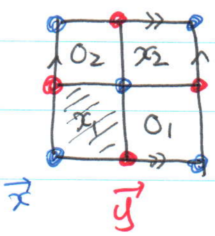
$\Rightarrow$ Alexander filtration on $(C(G), \partial)$

$\boxed{\text{full HFK}}$: filtered chain homotopy type of $(C(G), \partial)$

e.g. associated graded

$$CFK^- \rightsquigarrow HFK^-$$

<u>Example</u> unknot $n = 2$



$S(G) = \{\vec{x}, \vec{y}\}$

$A(\vec{x}) - A(\vec{y}) = 1$ $\qquad$ $\vec{X}$ has bigrading $(0,0)$

$M(\vec{x}) - M(\vec{y}) = 1$ $\qquad$ $\vec{y}$ $\quad$ " $\quad$ $(-1,-1)$

① C^- $\qquad$ $\mathbb{F}[U_1, U_2]\langle \vec{x}, \vec{y} \rangle$

$$\partial^- \vec{x} = 0$$

$$\partial^- \vec{y} = (U_1 - U_2)\vec{x}$$

$$HFK^-(\text{unknot}) = \mathbb{F}[U]/_{U = U_1 = U_2}$$

② $\widehat{C}$ set $U_1 = 0$ over $\mathbb{F}[U_2]$ $\quad \vec{y} \xrightarrow{U_2} \vec{x}$

$\widehat{HFK} = \mathbb{F}$ supported in degree $(0,0)$

$\chi(\widehat{HFK}) = 1 = \Delta_K(t)$

genus $(K) = 0$

fibered $\quad \operatorname{rk} \widehat{HFK}|_{A=0} = 1$

③ $\widetilde{C} = \mathbb{F}\langle \vec{x}, \vec{y} \rangle \quad \partial = 0$

$\widetilde{HFK} = \widetilde{C} = V \quad \operatorname{rk} V = 2$

$= \widehat{HFK} \otimes V$

④ $C(G) = \mathbb{F}[U_1, U_2]\langle \vec{x}, \vec{y} \rangle$

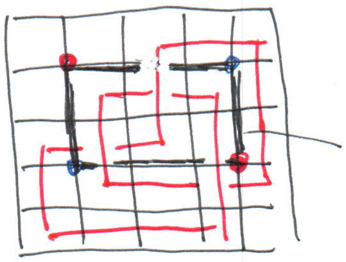
$\partial \vec{y} = (U_1 - U_2)\vec{x}$

$\partial \vec{x} = \vec{y} + \vec{y} = 0 \qquad H_*(C(G), \partial) = \mathbb{F}[U]$

<u>NB.</u> If we keep track of $\vec{x}$

$$\left.\begin{aligned} \partial \vec{x} &= (x_1 - x_2)\vec{y} \\ \partial \vec{y} &= (O_1 - O_2)\vec{y} \end{aligned}\right) \Rightarrow \partial^2 \neq 0$$

<u>More about the Alexander gradings</u>



$A(\vec{x}) - A(\vec{y}) = \sum x_i(r) - \sum O_i(r)$

$= \ell k(\partial r, K)$

linking #

$\because \quad \sum_i \underbrace{w(K, x_i) - w(K, y_i)}_{\text{winding \#}}$

$$A(\vec{x}) = \sum_i w(K, x_i) + \text{some constant}$$

depending only on the grid diagram

$$\chi(\widehat{HFK}) = \sum_{M,A} (-1)^M q^A \operatorname{rank} \widehat{HFK}_M(K, A)$$

$$= \Delta_K(q) \qquad \text{Alexander polynomial}$$

$$\chi(\widetilde{HFK}) = \chi(\widehat{HFK} \otimes V^{n-1}) = (1-q^{-1})^{n-1} \Delta_K(q)$$

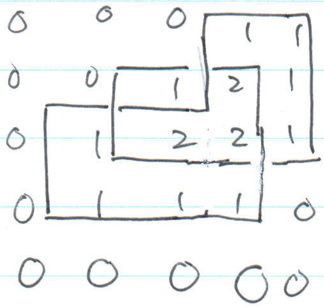
$$\approx (1-q)^{n-1} \Delta_K(q)$$

up to monomials

$$= \sum_{\vec{x} \in S(G)} \pm q^{A(\vec{x})}$$

$$\approx \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\, q^{w(K, x_{1,\sigma(1)})} \cdots q^{w(K, x_{n,\sigma(n)})}$$

$$= \det\left(q^{w(K, x_{ij})} \right)$$



$$\det \begin{bmatrix} 1 & 1 & 1 & q & q \\ 1 & 1 & q & q^2 & q \\ 1 & q & q^2 & q^2 & q \\ 1 & q & q & q & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \approx \Delta_K(q)(1-q)^{n-1}$$

new formula for the Alexander polynomial

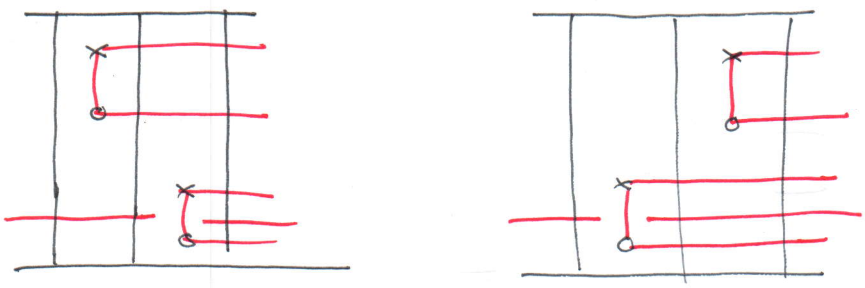
<u>Thm.</u> The filtered chain homotopy type of $(C(G), \partial)$ is a knot invariant.

(Hence $\widetilde{HFK}$, $\widehat{HFK}$: knot inv.)

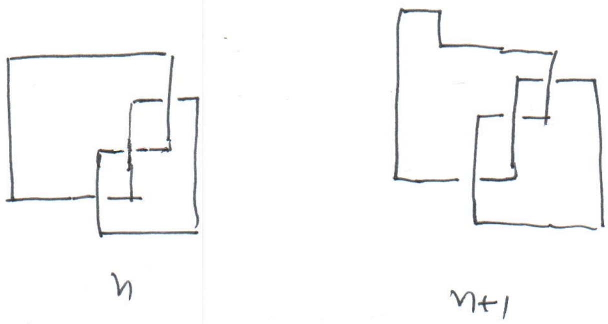
(Proof) Any two grid diagrams for the same knot K
are related by a sequence of Cromwell-Dynnikov moves

① cyclic permutation of rows or columns
by def. C(G) does not change
because we work on torus

② commutation of columns (or row)



③ stabilization



NB. ① change the crossing number